① $\forall v \in D : d^+(v) = 1 \quad |V(D)| \geq n$

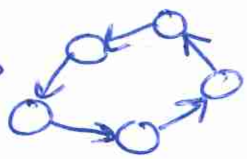
a) D is weakly connected

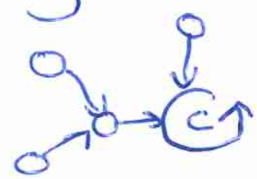
As shown in class, D must have a cycle if $\forall v \in D : d^+(v) \geq 1$

$\boxed{\text{minimum} = 1}$

Can D have more than one cycle?

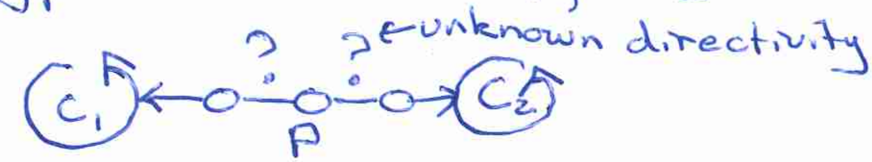
- We note that in some cycle C, there can be no out edges "exiting" the cycle from any $v \in C$

$C \Rightarrow$ 



$\Rightarrow$ Therefore, there can only be edges "entering" any given cycle $C \in D$

- In order to maintain weak connectivity with two or more cycles, there must be some set of edges between a hypothetical C_1, C_2 forming weak path P



- However, we observe if $|V(P)| = x$, then $|E(P)| = x + 1$, as the final edges point from P into C_1, C_2. As $d^+(v) = 1$, such P can't exist.

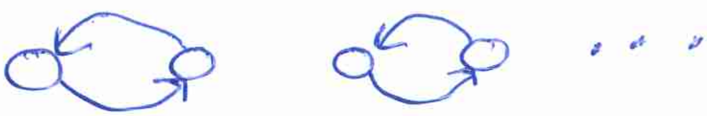
$\Rightarrow \boxed{\text{maximum} = 1}$

b) D is no longer weakly connected

- As we've just proved, the maximum number of cycles in a weak component is one

$\Rightarrow$ we want to maximize the number of weak components

- As we disallow self loops, the minimum size of a component is two



- So our maximum number of cycles is $\boxed{\left\lfloor \frac{|V(D)|}{2} \right\rfloor} = \left\lfloor \frac{n}{2} \right\rfloor$

c) We now allow self loops. We can use the same logic to get:

- Our maximum is now $\boxed{|V(D)|} = n$

② $S = \{1, 2, 1, 1, 4, 3\}$

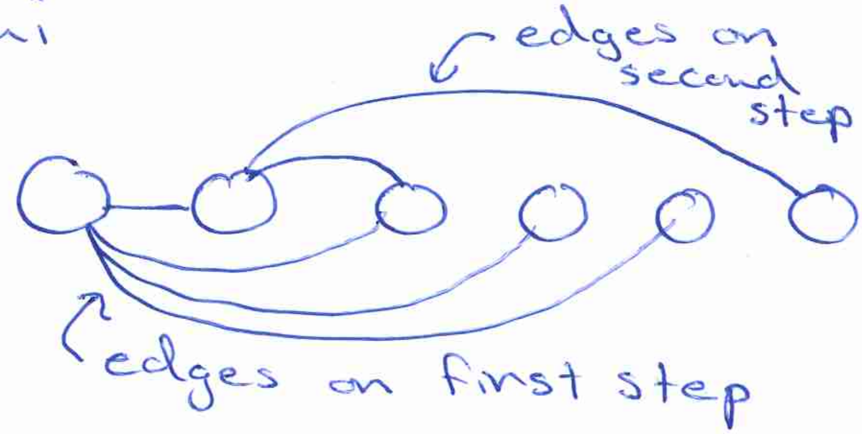
Using Havel-Hakimi

$\{4, 3, 2, 1, 1, 1\}$

$\quad -1\ -1\ -1\ -1$

$\{2, 1, 1\}$

$\quad -1\ -1$



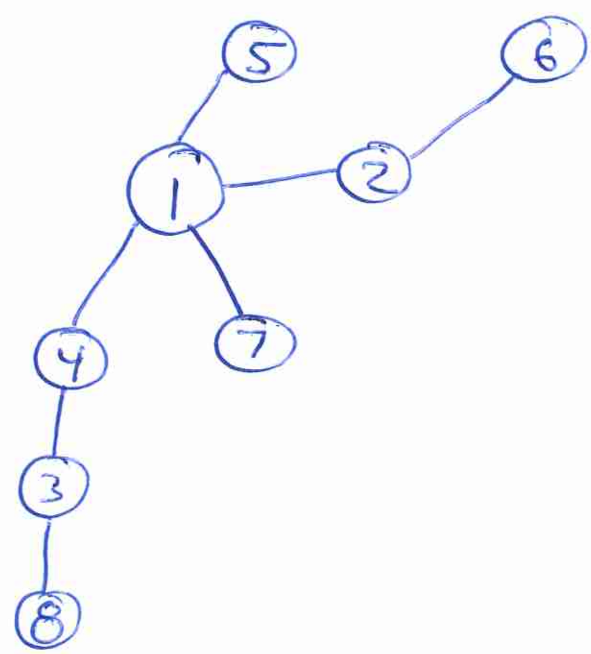
As a prüfer code

$S = \{\cancel{1}, \cancel{2}, \cancel{1}, \cancel{1}, \cancel{4}, \cancel{3}\}$

$V = \{\cancel{1}, \cancel{2}, 3, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7}, 8\}$

Edes = (5, 1)
(6, 2)
(2, 1)
(7, 1)
(1, 4)
(4, 3)
(3, 8)

$\Rightarrow$



(3) - We observe that the degree sequence is all even

- As we proved in class:

G is Eulerian iff $\forall v \in V(G) : d(v)$ is even assuming G is connected

$\Rightarrow$ Therefore, $\exists R \in G$ where R is a closed trail containing all $e \in E(G)$

- So for any $u, v \in V(G)$, there exists two paths P_1, P_2 connecting v and u

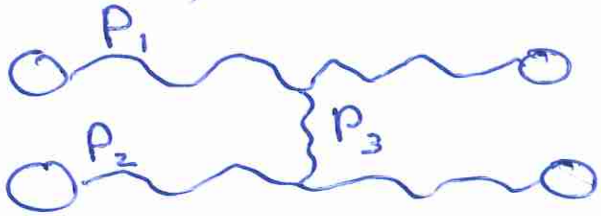
- Removal of any single edge will only disconnect P_1 or P_2 from u, v

$\Rightarrow$ So no single edge will disconnect G, or G has no cut edge $\square$

④ Note: this proof is valid for any connected graph, not only trees

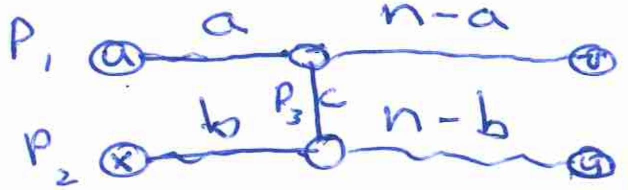
- We consider hypothetical tree T which contains $P_1, P_2 \in T$ where P_1, P_2 are of some maximum length n

- We consider the instance where no such $v \in V(T)$, $v \notin P_1, P_2$ exists
  → i.e. P_1, P_2 have no share vertex



⇒ So there must exist some minimum path connecting P_1, P_2 denoted as P_3
(T is connected, so such a P_3 must exist)

- Consider the below:



if $a > b$:
$|u,y\text{-path}| = a + n - b + c > n$
if $b > a$:
$|v,x\text{-path}| = b + n - a + c > n$
if $b = a \geq \frac{n}{2}$
$|u,x\text{-path}| = a + b + c > n$
if $b = a < \frac{n}{2}$
$|v,y\text{-path}| = n - a + n - b + c > n$

→ regardless of case, there always exists some larger path, a contradiction □

⑤ - Consider if G is connected, then it must have a minimum possible degree of 1 and a maximum possible degree of $n-1$

$\Rightarrow$ possible degrees of $\{1, 2, \dots, n-1\}$, where there are $n-1$ possible degrees for n vertices

$\Rightarrow$ by pigeon hole principle, at least one degree must be repeated ✓

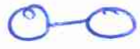
- Now consider if G is disconnected, then each component of G will have its own independent degree sequence

$\Rightarrow$ we can repeat the same argument above for each component of G $\square$

Note: we need to consider the disconnected case separately, otherwise we could include a vertex of degree zero, invalidating our first argument

(6) Show if $|V(G)| > |E(G)| \Rightarrow G$ must have at least one component that is a tree

We consider induction on edges of G

Base:  $\Rightarrow$ a single edge fits our assumptions and is a tree

Hypothesis: assume for some $P(k) = H$ s.t. $|V(H)| > |E(H)| \Rightarrow H$ has at least one tree component

Inductive Step: We construct H by contracting some edge $e \in E(G)$, $H = G \cdot e$, $|V(G)| > |E(G)|$

- We note $V(H) = V(G) - 1$, $E(H) = E(G) - 1$ fits our assumption, we make our I.H.
- We consider two cases:

Case 1: e was on a cycle. As we've seen, edge contraction retains cycles, so there is some other component in H and therefor G without a cycle $\Rightarrow$ tree

Case 2: e was not on a cycle. Edge contraction does not create cycles, so some tree component of $H \Rightarrow$ some tree component of G $\square$